# 2016

## MATHEMATICS

( Major )

Paper : 5.2

**( Topology )**

*Full Marks* : 60

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

**1.** Answer the following questions : 1×7=7

*(a)* Give an example to show that in a metric space, a Cauchy sequence may not always be convergent.

*(b)* Find the derived set, interior and closure of the set

$$A = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$$

in the real line $\mathbb{R}$ with the usual metric.

*(c)* For $x, y \in \mathbb{R}$, define $d(x, y) = | x^2 - y^2 |$. Examine whether $d$ is a metric on $\mathbb{R}$.

*(d)* Let $X = \{a, b, c\}$. Which of the following sets is not a topology on $X$?

*(i)* $\{\phi, X\}$

*(ii)* $\{\phi, \{a\}, X\}$

*(iii)* $\{\phi, \{a\}, \{b\}, \{a, b\}, X\}$

*(iv)* $\{\phi, \{a\}, \{b\}, \{c\}, X\}$

*(e)* Let $X$ be any set and let $\mathscr{T} = \{\phi, A, B, X\}$, where $A$ and $B$ are non-empty disjoint proper subsets of $X$. Find the conditions $A$ and $B$ must satisfy in order that $\mathscr{T}$ will be a topology on $X$.

*(f)* Show that the usual metric on $\mathbb{Z}$ (the set of all integers) induces the discrete topology for $\mathbb{Z}$.

*(g)* Define Hilbert space and give an example.

**2.** Answer the following questions : 2×4=8

*(a)* Every subset of a discrete metric space is closed. Justify whether it is true or false.

*(b)* Let $X = \{a, b, c, d, e\}$ and let $S = \{\{a, b\}, \{b, c\}, \{a, d, e\}\}$. Find the topology on $X$ generated by $S$.

*(c)* Show that every normed linear space is a metric space.

*(d)* Prove the parallelogram law in an inner product space $(X, <\cdot, \cdot>)$.





**3.** Answer the following questions : 5×3=15

*(a)* Let $(X, d)$ be a complete metric space and let $Y$ be a subspace of $X$. Prove that $Y$ is complete if and only if $Y$ is closed.

*(b)* Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. Show that $A$ is closed if and only if $A$ contains each of its limit points.

*Or*

Prove that a mapping $f$ from a topological space $X$ into another topological space $Y$ is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for every set $A \subset X$.

*(c)* Show that $\mathbb{C}^n$ is a Banach space.

*Or*

Let $(X, <\cdot, \cdot>)$ be an inner product space. Prove that for all $x, y \in X$,

$$4\langle x, y\rangle = ||x+y||^2 - ||x-y||^2 + i\,||x+iy||^2 - i\,||x-iy||^2$$

**4.** Answer the following questions : 10×3=30

*(a)* Prove that the set $\mathbb{R}^n$ of $n$-tuples $x = (x_1, x_2, \cdots, x_n)$ of real numbers is a complete metric space with respect to the usual metric. 10

*Or*

State and prove Cantor's intersection theorem for metric spaces. 10

*(b)* Prove that in a metric space $(X, d)$, the union of a finite number of nowhere dense sets is nowhere dense. Again, if $A$ is nowhere dense in $X$, then show that each open sphere in $X$ contains a closed sphere which contains no point of $A$. 6+4=10

*Or*

Let $X$ be a metric space and $Y$ be a complete metric space. Let $A$ be a dense subspace of $X$. If $f: A \to Y$ is uniformly continuous, then prove that $f$ can be extended uniquely to a uniformly continuous mapping $g: X \to Y$. 10

*(c)* Prove that a subset $A$ of $\mathbb{R}$ is compact if and only if it is closed and bounded. 10

*Or*

Prove that the continuous image of a connected metric space is connected. Also show that the range of a continuous real-valued function defined on a connected space is an interval. 10

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