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dekaBipul

Solution to the Θ- equation

The θ- equation is

$$\frac{1}{\Theta}\left[\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)\right] + l(l+1)\sin^2\theta = m^2$$

$$\Rightarrow \sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left\{l(l+1)\sin^2\theta - m^2\right\}\Theta = 0 \longrightarrow (1)$$

Solution of equation is

$$\Theta(\theta) = A\,P_l^m(\cos\theta) \quad , \quad A = \text{constant}.$$

$$P_l^m(\cos\theta) = (1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|} P_l(\cos\theta) \longrightarrow (2)$$

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l (x^2-1)^l \longrightarrow (3)$$

Thus the solution of the angular part of the SE for H-atom

$$Y_l^m(\theta,\phi) = (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}\, e^{im\phi}\, P_l^m(\cos\theta) \qquad \left[\therefore Y_l^m(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)\right]$$

—x—

Identification of the constant 'l'

We have not identified l in the separation constant $l(l+1)$. Let's identify it and find out its true nature.

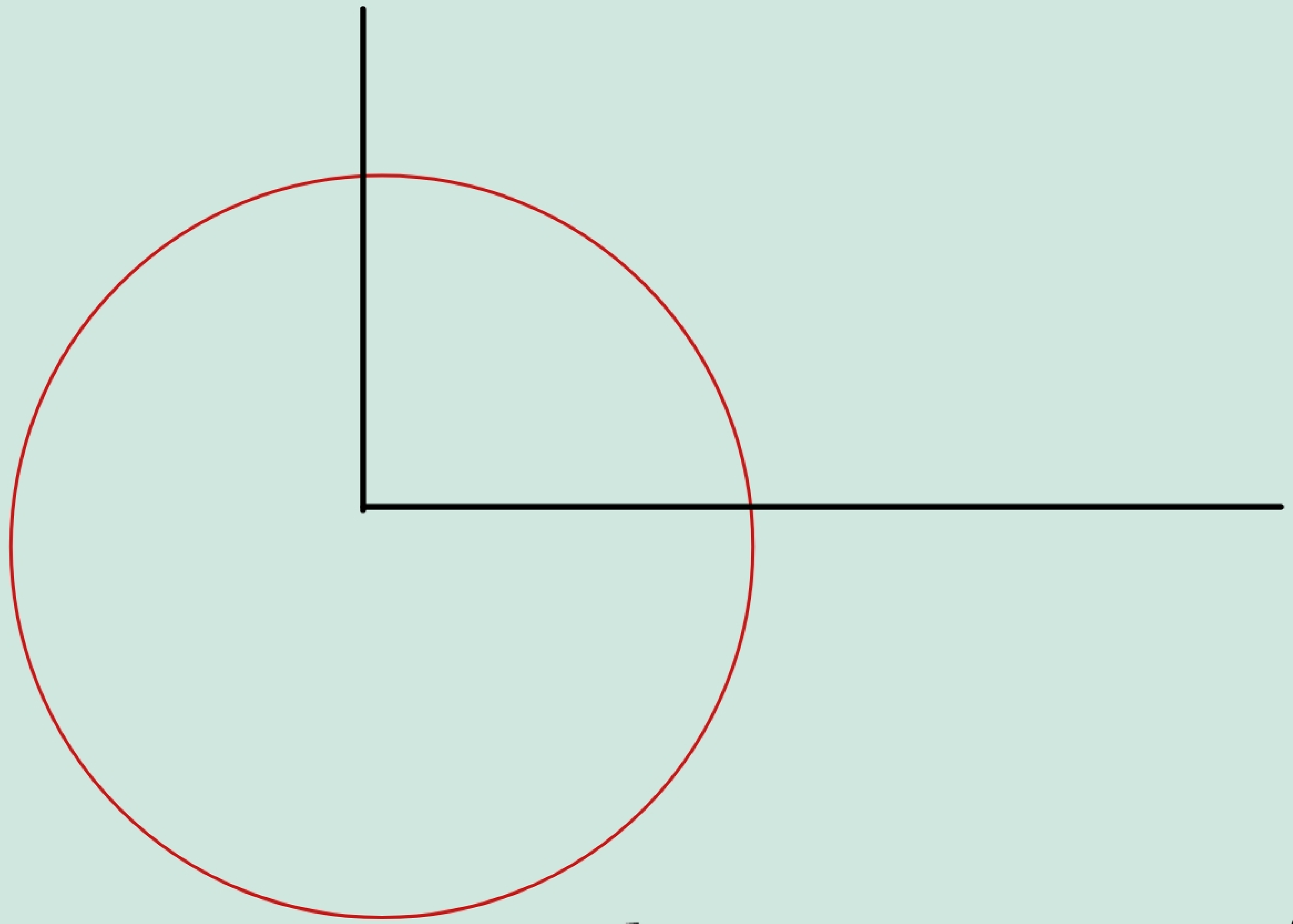
We know 'm' can have values, $0, \pm 1, \pm 2, \pm 3 \ldots$ etc. For $m = 1$ and $l = 0$ let's calculate $P_l^m(\cos\theta)$. It can readily be found out that from equation (2) and (3) that a combination of m & l such that $m > l$ is not valid in our context. So any value of m must be such that $m \leq l$. And for, eg, $l = 1$, we have $m = 0, \pm 1$, which satisfies the condition $m \leq l$. Thus for every l there are $(2l+1)$ values of m. This indicates that m and l are playing the seme role that magnetic & orbital quantum number plays in Bohr's atomic model. Thus 'm' and 'l' appearing in the constant m^2 & $l(l+1)$ are indeed magnetic & orbital quantum number only. This is the beauty of SE which shows that quantization is a natural process (l can have only non-negative integer value)

Calculation of $Y_l^m(\theta, \phi)$

Let, $l = 1$, then $m = 0$ (this is s- orbital)

So, $$Y_l^m = Y_1^0 = \frac{1}{\sqrt{4\pi}} P_1^0(\cos\theta) = \sqrt{\frac{1}{4\pi}}$$

Y_1^0 is independent of θ & ϕ. The shape of Y_1^0



This is S-orbital, which is Spherically Symmetric.