2016

MATHEMATICS

(Major)

Paper : 1.1

(Algebra and Trigonometry)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

1. Answer/Choose the correct option : 1×10=10

(a) Let $A = \{1, 2, 3, 4\}$. Give an example of a relation in A which is transitive but not reflexive or symmetric.

(b) Which statement is correct?

(i) $f : R \to R$ given by $f(x) = x^2$ is injective.

(ii) $f : N \to N$ given by $f(x) = 2x$ is surjective.

(iii) $f: N \to E$ given by $f(x) = 2x$ is not surjective. (E is the set of non-negative even integers)

(iv) $f: N \to N$ given by $f(x) = x^2$ is injective.

(c) If m, n and x are three elements of a group and $mnxnm = y$, then

(i) $x = n^{-1}m^{-1}ym^{-1}n^{-1}$

(ii) $x = nm^{-1}ym^{-1}n$

(iii) $x = m^{-1}n^{-1}yn^{-1}m^{-1}$

(iv) $x = m^{-1}n^{-1}ym^{-1}n^{-1}$

(d) For the group $\langle Z, +\rangle$ and normal subgroup $N = \{3n | n \in Z\}$, what is the order of the quotient group $\frac{Z}{N}$?

(e) If n is an integer, then $(1+i)^n + (1-i)^n$ equals

(i) $2^{\frac{n}{2}+1} \cos\frac{n\pi}{4}$

(ii) $2^{\frac{n}{2}} \cos\frac{n\pi}{4}$

(iii) $2^{\frac{2}{n}-1} \cos\frac{\pi}{4}$

(iv) $2^{\frac{n}{2}} \sin\frac{n\pi}{4}$

(f) Find out the correct statement(s) :

(i) $(1+\omega+\omega^2)^3 - (1-\omega+\omega^2)^3 = -1$

(ii) $(1+\omega+\omega^2)^3 - (1-\omega+\omega^2)^3 = 1$

(iii) $(1+\omega+\omega^2)^3 - (1-\omega+\omega^2)^3 = 0$

(iv) $(1+\omega+\omega^2)^3 - (1-\omega+\omega^2)^3 = \frac{i+\sqrt{3}}{2}$

(g) If α, β, γ, δ are the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$, then the value of $\Sigma\alpha^2\beta\gamma$ is

(i) $-pq + 3r$

(ii) $pr - 4s$

(iii) $q^2 - 2pr + 2s$

(iv) $-r/s$

(h) The rank of the matrix

$$A = \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$$

is

(i) 1

(ii) 0

(iii) 2

(iv) 3

(i) If A is any square matrix, then find the correct statement(s) :

(i) $A + A'$ is not symmetric

(ii) $A - A'$ is symmetric

(iii) $A - A'$ is skew-symmetric

(iv) $A + A'$ is skew-symmetric

(j) A matrix is idempotent if $A^2 = A$. If $AB = A$ and $BA = B$, then show that A is idempotent.

2. Give answers to the following questions : 2×5=10

(a) With an example show that we can have maps f and g such that $g \circ f$ is one-one and onto but f need not be onto and g need not be one-one.

(b) Let A and B be two square matrices of same order. If $AB = I$, prove that $BA = I$.

(c) Find the centre of S_3 where $S = \{1, 2, 3\}$.

(d) If $\sin(\alpha + i\beta) = x + iy$, prove that

$$x^2 \operatorname{cosec}^2 \alpha - y^2 \sec^2 \alpha = 1$$

(e) How many complex roots does the equation $x^4 + 2x^2 + 3x - 1 = 0$ have? Apply Descartes' rule of signs for finding the complex roots.

3. Answer any *four* parts : 5×4=20

(a) Let $f : A \to B$ and $g : B \to C$ be one-to-one and onto functions. Show that $(g \circ f)^{-1}$ exists and $(g \circ f)^{-1} = f^{-1} \circ g^{-1} : C \to A$.

(b) If H and K are finite subgroups of G of order $O(H)$ and $O(K)$ respectively, then prove that

$$O(HK) = \frac{O(H)O(K)}{O(H \cap K)}$$

(c) Prove that a subgroup H of a group G is a normal subgroup of G if and only if the product of two right cosets is again a right coset of H in G.

(d) Deduce the following series :

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots + \frac{(-1)^{n-1} x^{2n-1}}{2n-1} + \cdots$$

(e) If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$, then find the equation whose roots are $\alpha\beta + \beta\gamma$, $\beta\gamma + \gamma\alpha$, $\gamma\alpha + \alpha\beta$.

(f) Reduce the following matrix to normal form and find its rank :

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

4. Answer any *one* part : 10

(a) *(i)* Show that a subgroup of index 2 in a group G is a normal subgroup of G.

(ii) In a set of n elements define S_n and A_n where the symbols have their usual meanings.

(iii) Show by example that a quotient group may be Abelian but parent group of the quotient group may not be Abelian. 4+2+4

(b) *(i)* Show that an infinite cyclic group has precisely 2 generators.

(ii) Show that a group G of prime order cannot have non-trivial subgroups.

(iii) Let a, n $(n \geq 1)$ be any integers s.t. g.c.d. $(a, n) = 1$. Prove that

$$a^{\phi(n)} \equiv 1 (\bmod n)$$ 4+2+4

5. Answer any *one* part : 10

(a) *(i)* Show that the roots of the equation $Z^n = (Z+1)^n$ where n is a positive integer >1 are collinear points in the Z plane.

(ii) Using De Moivre's theorem, solve

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$$

(iii) If $\sin(\theta + i\phi) = \tan(x + iy)$, then show that $\dfrac{\tan\theta}{\tan\phi} = \dfrac{\sin 2x}{\sinh 2y}$. 3+4+3

(b) *(i)* Expand $\sin^4\theta\cos^2\theta$ in a series of cosines of multiples of θ.

(ii) If $x < (\sqrt{2} - 1)$, then prove that

$$2\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots\right) = \frac{2x}{1-x^2} - \frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3 + \frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5 - \cdots$$

(iii) Show that

$$\tan\left(i\log\frac{a-ib}{a+ib}\right) = \frac{2ab}{a^2 - b^2}$$ 3+4+3

6. Answer any *two* parts : 5×2=10

(a) If $\alpha, \beta, \gamma, \delta$ are roots of the biquadratic equation $x^4 + px^3 + qx^2 + rx + \delta = 0$, find the value of $\Sigma\alpha^2\beta\gamma$ and $\Sigma\alpha^2\beta^2$.

(b) Solve by Cardan's method :

$$x^3 - 6x^2 - 6x - 7 = 0$$

(c) Find the equation whose roots are squares of the differences of the roots of the equation $x^3 + x + 2 = 0$ and deduce from the resulting equation the nature of the roots of the given cubic.

(d) Solve the equation

$$16x^3 - 44x^2 + 36x - 9 = 0,$$

given that roots are in harmonic progression.

7. Answer any *two* parts : 5×2=10

(a) Define skew Hermitian matrix. Prove that every Harmitian matrix can be written as $A = B + iC$, where B is real and symmetric and C is real and skew-symmetric.

(b) *(i)* If a non-singular matrix A is symmetric, prove that A^{-1} is also symmetric. 2

(ii) If A is a $n \times n$ non-singular matrix, then prove that $(A')^{-1} = (A^{-1})'$. 3

(c) Investigate for what values of a and b the simulteneous equations

$$x_1 + x_2 + x_3 = 6$$

$$x_1 + 2x_2 + 3x_3 = 10$$

$$x_1 + 2x_2 + ax_3 = b$$

have

(i) no solution;

(ii) an unique solution;

(iii) an infinite number of solutions.

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